Department of Mechanical and Industrial Engineering

ME 3455: Dynamics and Vibrations (Fall 2017)

Term Project 1: Dynamic Modeling and Experimental Verification of a Gantry System

Submitted by

Group 14

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Abstract

Gantry systems are used across the industrial world for a variety of applications. In this experiment, a gantry system's dynamic motions along a linear rail were analyzed and a relationship was found connecting the input voltage to the linear and angular displacements of the pendulum. The theoretical relationship was then compared with empirical data. It was found that these two experimental results were very similar, resulting in a mean position error of 0.0043 m and a mean angle error of 0.0112 radians.

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Lab Section:	**Section 4**
Course Instructor:	**Prof. Jalili**

List of Symbols

m_o	Mass of the cart
m_p	Mass of pendulum
O	Center of gravity of cart
P	Center of gravity of pendulum
G	Acceleration due to gravity
X	Position of cart
$\dot{x}$	Velocity of cart
$\ddot{x}$	Acceleration of cart
θ	Angular position of pendulum
$\dot{\theta}$	Angular velocity of pendulum
$\ddot{\theta}$	Angular acceleration of pendulum
L	Length of pendulum
F	Input force
K_g	Internal gear ratio between gear and shaft
K_t	Motor torque constant
K_m	Back emf constant
r_g	Radius of cart gear
V_{in}	Voltage input
I	Current following through motor
R	Motor armature resistance

Introduction

Gantry systems are a type of control and manufacturing problem. They are widely used in manufacturing to carry a tool from one end of the production line to the other with little vibration. To be able to build a control system to move the position of the cart as fast possible, the equations of motion need to be derived to model the cart's movement with pendulum.

The equations of motion are used to understand how a mass connected to a cart behaves as it travels down a linear track and to see how the position, velocity, and acceleration of the cart affect the pendulum movement. Using the derived equations of motion for a simple gantry system, these equations can be modeled and verified from experimental data using Simulink, an application within MATLAB.

Figure 1: Overhead crane used in industrial application

An example of an industrial gantry system in use is an overhead crane as shown in Figure 1 above. Overhead cranes are used to move a load side to side or backward and forward on a linear track. Like the simple gantry studied in this project, the overhead crane has a cart that moves on a linear track. This system is very useful for moving heavy loads and components overhead in manufacturing or assembly facilities through production. The pendulum in the simplified gantry system resembles the suspended cable to carry mass or tool.

Experimental Setup

The experimental setup is similar to that found in industrial settings but on a much smaller scale. A uniform pendulum is attached to a cart on a linear rail. A DC motor powers the drive shaft, providing movement through the gear and linear rail. A separate wheel connects to an internal encoder which provides information on the position of the cart on the rail. As the cart moves along the rail, the pendulum will deflect back due to inertia from the cart. This angular deflection is again captured by an encoder located at the attachment of the pendulum. An image of the setup can be found below in Figure 2.

Figure 2: Gantry experimental setup

When the motor receives an input current (*i(t)*), the resultant torque is translated into a linear force (*F*) on the rail, which results in the cart's motion. Equations 1 & 2 relate the linear force to the current and the current to the voltage respectively.

$$F(t) = \frac{k_g k_t}{r_g} i(t) \quad (1)$$

$$i(t) = \frac{V_{in}(t)}{R} - \frac{k_m k_g \dot{x}_1(t)}{R r_g} \quad (2)$$

The force is based on the gear ratio (k_g) and radius of the gear (r_g) as well as the torque constant (k_t) for the DC motor. Current is based on the internal resistance (R) of the motor as well as the back electromotive force (emf) produced by the motor. This last term is ultimately dependent on the linear velocity ($\dot{x}_1$) and the back emf constant (k_m). Values of all the constants can be found in Appendix C.

The goal of this experiment is to determine the cart displacement (*x(t)*) and pendulum angular displacement (θ(t)) in relation to the input voltage ($V_{in}(t)$). To begin, the group established two equations of motion for *x* and θ. The derivations began with the summation of forces and movements for two separate Free-Body-Diagrams (see Figure 3 & 4) and can be seen in Equations 3, 4 & 5 below. Step by step derivations can be seen in Appendix A and the final equations of motion are posted below in Equation 6 & 7 based on small angle approximations.

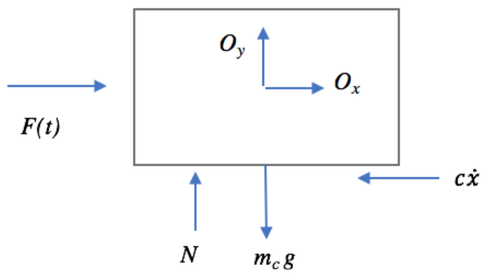


Figure 3: Free-Body-Diagram of cart

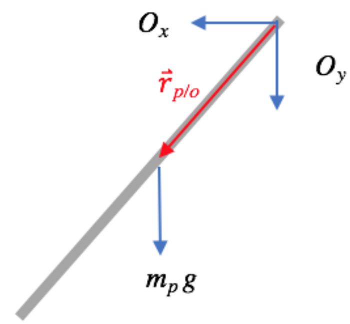


Figure 4: Free-Body-Diagram of pendulum

$$\Sigma F_x = F(t) + O_x - c\dot{x} = m_c \ddot{x}_{o_x} \tag{3}$$

$$\Sigma F_x = O_x = m_p \ddot{x}_{p_x} \tag{4}$$

$$\sum M_p = I_p \ddot{\theta} \tag{5}$$

The final equations of motion:

$$F(t) = (m_c + m_p)\ddot{x}_0 + c\dot{x} + \frac{1}{2} m_p l \ddot{\theta} \tag{6}$$

$$\frac{l}{3}\ddot{\theta} - \frac{1}{2}\ddot{x}_0 + \frac{1}{2} g\theta = 0 \tag{7}$$

The team then utilized Simulink to analytically solve the system of two differential equations (models can be seen in Appendix B). The team found angular and cart displacement vs time with a specific frequency and a damping constant which will be discussed in greater detail in the following section. The goal through this modelling was to numerically compare the theoretical results against empirical data.

Results

The experimental setup was tested 4 times, with input voltage sine waves at the following frequencies: π, 6π, 10π, and 20π. In each case, the signal amplitude was set to 3 Volts. The Simulink model was also run with this simulated input. The results of these tests are shown in Figures 5-8.

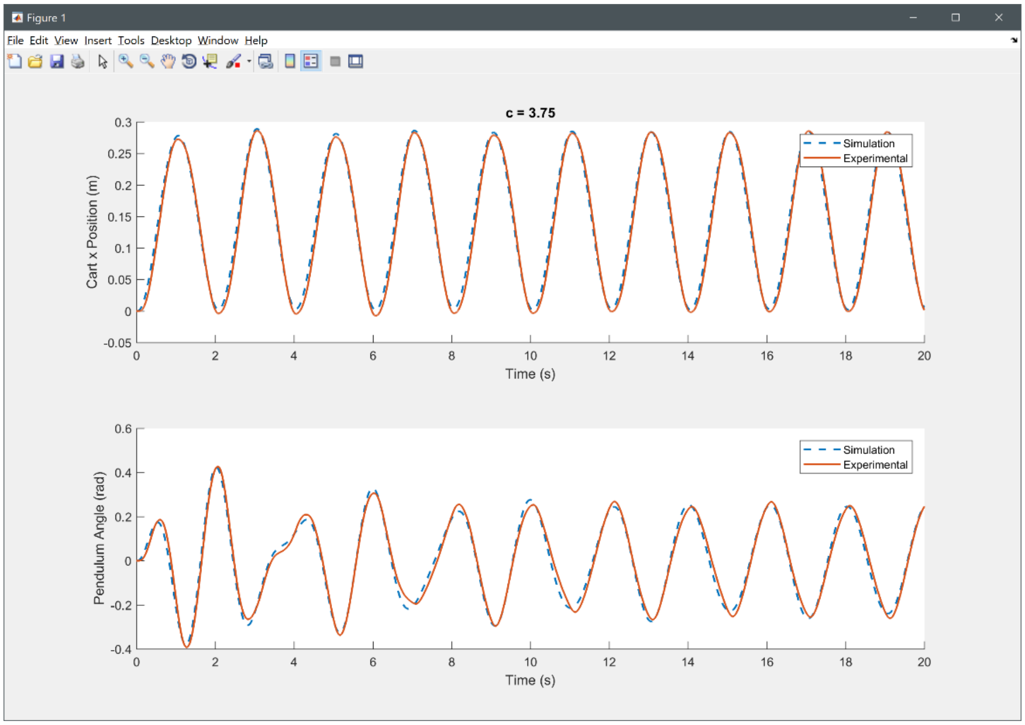


Figure 5: Position and Angle Waveforms, f = π, c = 3.75

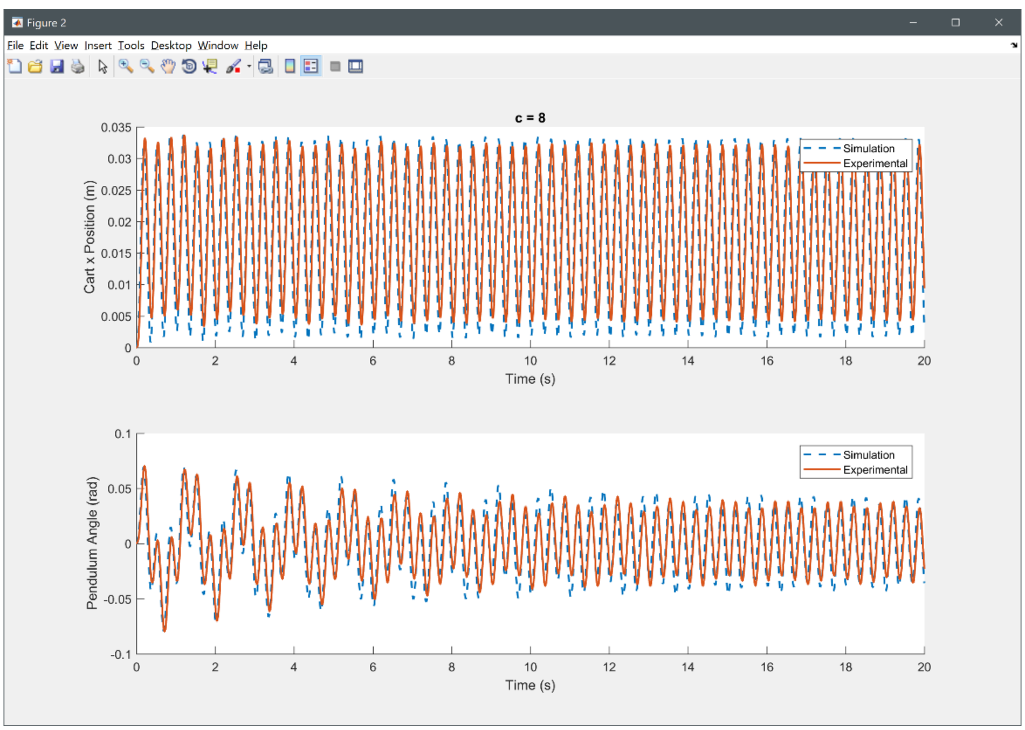


Figure 6: Position and Angle Waveforms, f = 6π, c = 8

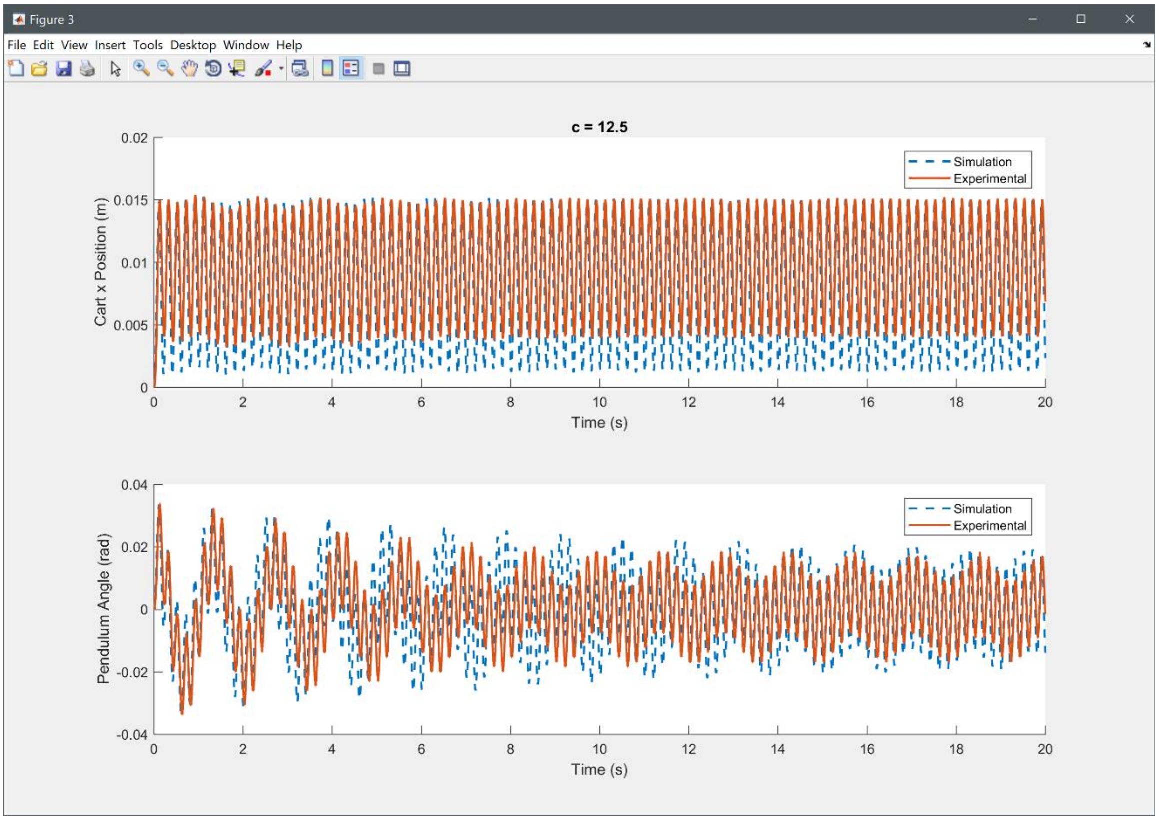


Figure 7: Position and Angle Waveforms, f = 10π, c = 12.5

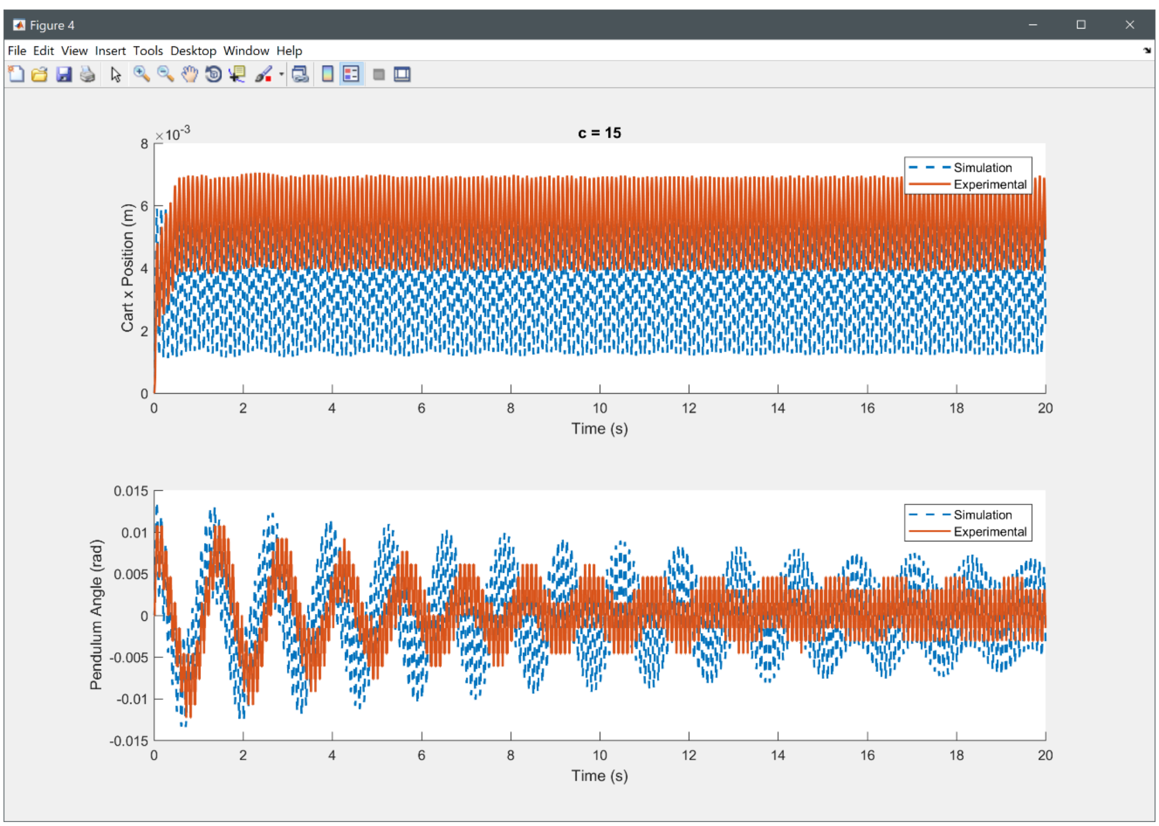


Figure 8: Position and Angle Waveforms, f = 20π, c = 15

It is evident from these plots that the Simulink model fits the experimental data well, especially in the case of low frequencies. In order to improve the fit, a damping constant was added (Equation 8). This damping constant was manually adjusted using a trial-and-error method to improve the fit of the Simulink model. It was noted, however, that the most accurate damping constant varied with input frequency. Since this constant should not need to vary, this indicates that the experimental setup was experiencing some damping forces other than the simple viscous damping of the theoretical model.

Conclusions

In order to evaluate the accuracy of the Simulink model, the error was calculated using the following formula:

$$\text{error} = \text{mean}(\text{abs}(x_{sim} - x_{exp}))$$

The error was determined to be:

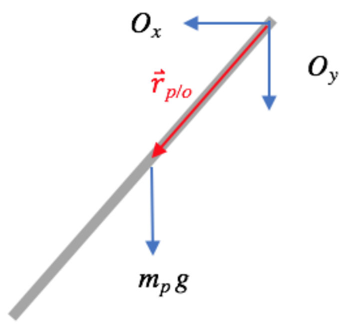
Frequency	Mean Position Error (m)	Mean Angle Error (rad)
pi	0.0089	0.0217
6pi	0.0043	0.0122
10pi	0.0022	0.0074
20pi	0.0018	0.0035
Mean	0.0043	0.0112

This error was likely caused by experimental conditions not included in the model, such as air resistance in the pendulum and friction in the encoders and bearings. As discussed previously, the manually determined damping constants indicate this as they vary with frequency. It could also have been caused by errors and noise in the sensing equipment. It is important to note that when the input voltage is 0, theoretically the cart should move freely. In reality, the motion of the cart creates a back current in the motor which impedes it. It is evident, however, that with an average error of 4.3 mm in cart position and 0.01 radians in pendulum angle, the Simulink model very closely approximated the experimental data.

Appendix A: Derivation of Equations of Motion

Pendulum:

FBD of Cart:



$$\Sigma F_y = -O_y - m_p g = m_p \ddot{x}_{p_y}$$
$$\ddot{x}_p = \ddot{x}_o + \ddot{\theta} \times \vec{r}_{p/o} - \dot{\theta}^2 \vec{r}_{p/o}$$
$$\vec{r}_{p/o} = (-\frac{1}{2} l \sin\theta\, \hat{\imath} - \frac{1}{2} l \cos\theta\, \hat{\jmath})$$
$$\ddot{x}_p = \ddot{x}_o \hat{\imath} + \ddot{\theta}\hat{k} \times (-\frac{1}{2} l \sin\theta\, \hat{\imath} - \frac{1}{2} l \cos\theta\, \hat{\jmath}) - \dot{\theta}^2(-\frac{1}{2} l \sin\theta\, \hat{\imath} - \frac{1}{2} l \cos\theta\, \hat{\jmath})$$
$$\ddot{x}_p = \ddot{x}_o \hat{\imath} - \ddot{\theta}\frac{1}{2} l \sin\theta\, \hat{\jmath} - \ddot{\theta}\frac{1}{2} l \cos\theta\, \hat{\imath} + \dot{\theta}^2 \frac{1}{2} l \sin\theta\, \hat{\imath} - \dot{\theta}^2 \frac{1}{2} l \cos\theta\, \hat{\jmath}$$
$$\ddot{x}_{p,y} = -\ddot{\theta}\frac{1}{2} l \sin\theta\, \hat{\jmath} - \dot{\theta}^2 \frac{1}{2} l \cos\theta\, \hat{\jmath}$$
$$\ddot{x}_{p,x} = \ddot{x}_o \hat{\imath} - \ddot{\theta}\frac{1}{2} l \cos\theta\, \hat{\imath} + \dot{\theta}^2 \frac{1}{2} l \sin\theta\, \hat{\imath}$$

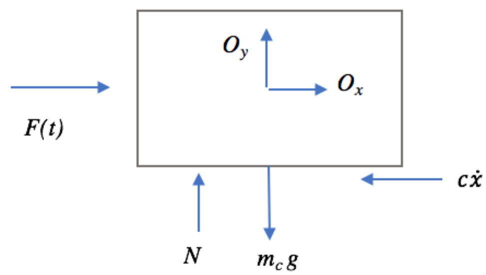
$$\Sigma F_x = -O_x = m_p \ddot{x}_{p_x}$$
$$O_x = m_p(\ddot{x}_o \hat{\imath} - \ddot{\theta}\tfrac{1}{2} l \cos\theta\, \hat{\imath} + \dot{\theta}^2 \tfrac{1}{2} l \sin\theta\, \hat{\imath})$$

$$\sum M_o = I_o \ddot{\theta} + \vec{r}_{p/o} \times m_p \ddot{x}_o$$
$$m_p g \frac{l}{2} \cos\theta \hat{k} = -\frac{1}{3} m_p l^2 \ddot{\theta} \hat{k} + \left(-\frac{1}{2} l \sin\theta\, \hat{\imath} - \frac{1}{2} l \cos\theta\, \hat{\jmath}\right) \times m_p \ddot{x}_o \hat{\imath}$$
$$m_p g \frac{l}{2} \cos\theta \hat{k} = -\frac{1}{3} m_p l^2 \ddot{\theta} \hat{k} - \frac{1}{2} m_p l\, \ddot{x}_o \cos\theta$$
$$\frac{l}{3}\ddot{\theta} - \frac{1}{2}\ddot{x}_0 \cos\theta + \frac{1}{2} g \sin\theta = 0$$

Equation A-1: $\frac{l}{3}\ddot{\theta} - \frac{1}{2}\ddot{x}_0 \cos\theta + \frac{1}{2} g \sin\theta = 0$

Cart:

FBD of Cart:



$$\Sigma F_x = F(t) + O_x - c\dot{x} = m_c \ddot{x}_{o_x}$$

$$F(t) + m_p(\ddot{x}_o - \ddot{\theta}\frac{1}{2}l\cos\theta + \dot{\theta}^2\frac{1}{2}l\sin\theta) - c\dot{x} = m_c\ddot{x}_{o_x}$$

$$F(t) = m_c\ddot{x}_0 + c\dot{x} - m_p(\ddot{x}_0 - \frac{1}{2}l\ddot{\theta}\cos\theta + \frac{1}{2}l\dot{\theta}^2\sin\theta)$$

Equation A-2: $F(t) = (m_c + m_p)\ddot{x}_0 + c\dot{x} + \frac{1}{2}m_p l\ddot{\theta}\cos\theta - \frac{1}{2}m_p l\dot{\theta}^2\sin\theta$

Input Force Equations (given):

$$F(t) = \frac{k_g k_t}{r_g}i(t)$$

$$i(t) = \frac{V_{in}(t)}{R} - \frac{k_m k_g \dot{x}_1(t)}{R r_g}$$

Small angle assumption where $\cos\theta = 1, \sin\theta = \theta$ for Equation A-1 and A-2

Equation A-3: $(m_c + m_p)\ddot{x}_0 - \frac{1}{2}m_p l\ddot{\theta} = F(t)$

$$F(t) = \frac{K_g K_t}{r_g R}V_{in}(t) - \frac{{K_g}^2 K_t K_m}{{r_g}^2 R}\dot{x}(t)$$

Equation A-4: $\frac{l}{3}\ddot{\theta} - \frac{1}{2}\ddot{x}_0 + \frac{1}{2}g\theta = 0$

Putting equations A-3 and A-4 in terms of only $\ddot{x}, \ddot{\theta}$

Equation A-5: $\ddot{x} = \frac{1}{\left[\frac{1}{4}m_p + m_c\right]}(F(t) - c\dot{x} - \frac{3}{4}m_{p}g\theta)$

Equation A-6: $\ddot{\theta} = \left[\frac{3}{2l - \frac{3(m_p l(m_p + m_c))}{2}}\right][(m_p + m_c)F(t) - (m_p + m_c c\dot{x} - g\theta)]$

Appendix B: Simulink Model

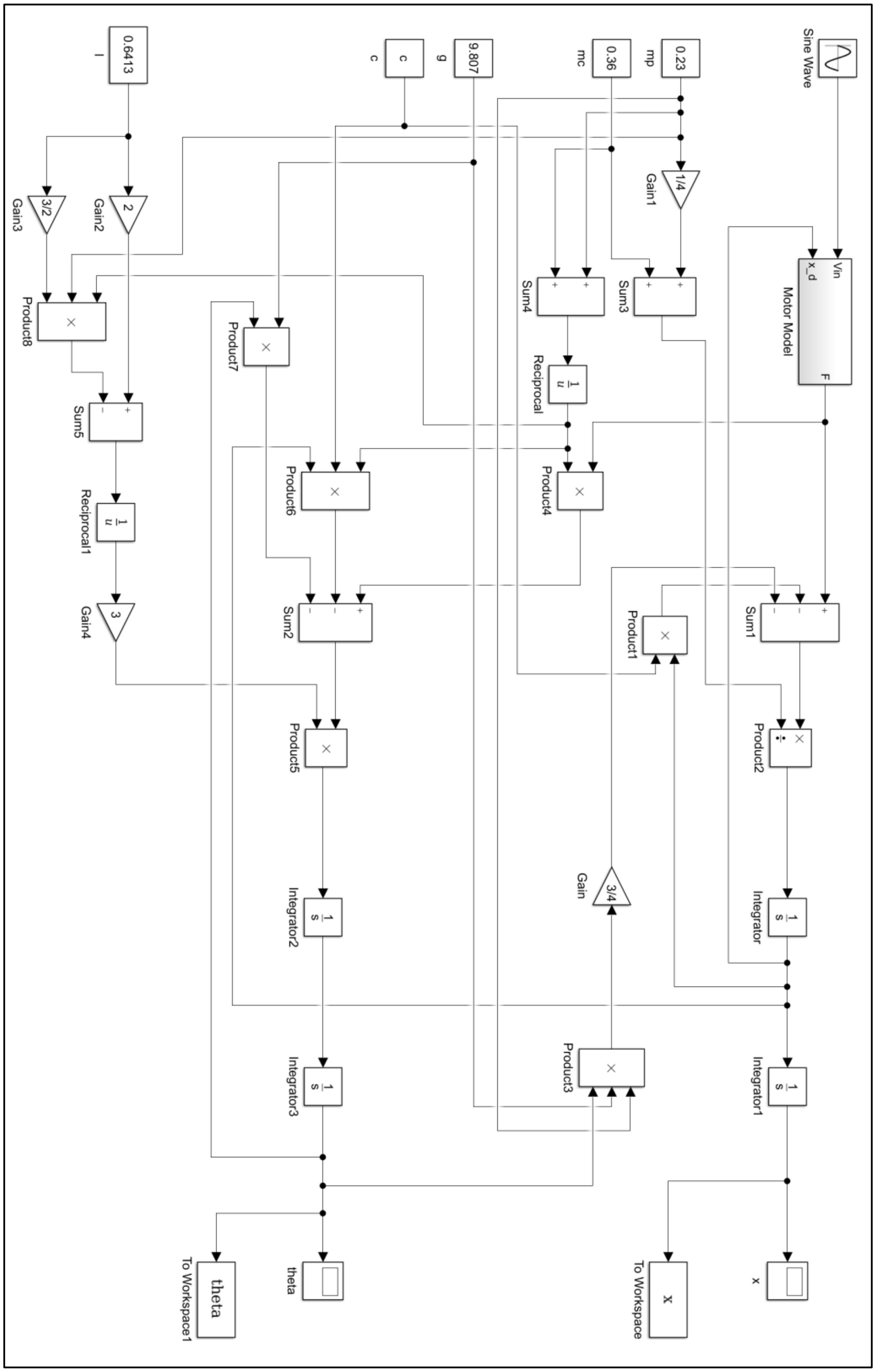


Figure 9: Cart-Pendulum Simulink Model

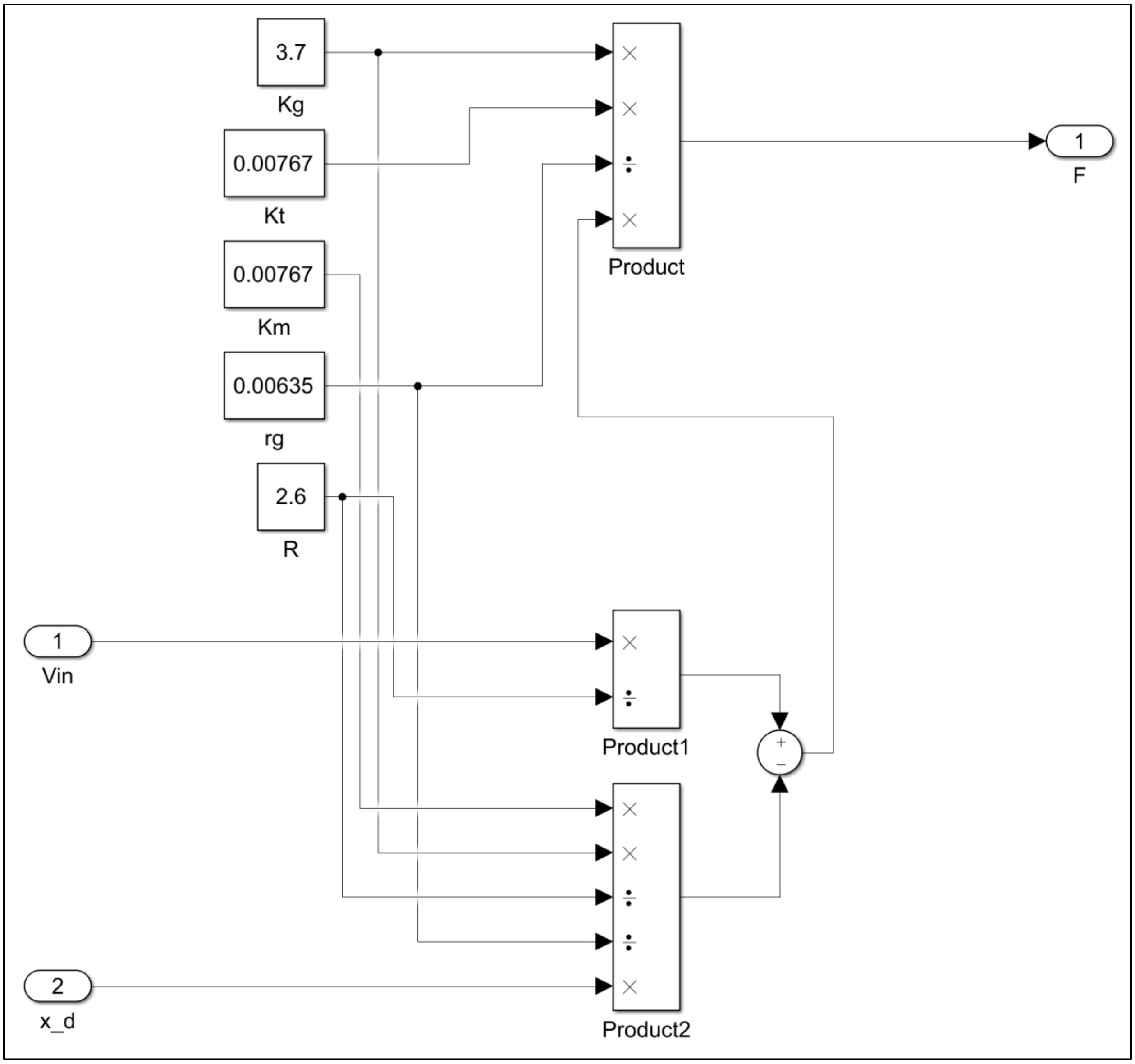


Figure 10: Drive Motor Simulink Model

Appendix C: System Parameters for Gantry System

m_o	0.360 kg
m_p	0.230 kg
g	9.81 m/s^2
l	0.6413 m
K_g	3.7:1
K_t	0.00767 Nm/Amp
K_m	0.00767 V.sec/rad
r_g	0.00635 m
V_{in}	3 Volts
R	2.6 Ω

Appendix D: MATLAB Data Analysis Code

```
error = struct('x', [], 'theta', []);

%%%%%%%%%%%%%%%%%%%%%%%%

f = pi;
c = 3.75;

sim('gantry')

hFig = figure(1);
set(hFig, 'Position', [0 0 1100 700]);

subplot(2,1,1);
hold on;
plot(x, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), Gantrypirad(:, 2), 'LineWidth', 1.5);
title('c = 3.75');
xlabel('Time (s)');
ylabel('Cart x Position (m)');
legend('Simulation', 'Experimental');

subplot(2,1,2);
hold on;
plot(theta, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), -1 .* Gantrypirad(:, 3), 'LineWidth', 1.5);
title('');
xlabel('Time (s)');
ylabel('Pendulum Angle (rad)');
legend('Simulation', 'Experimental');

error.x = [error.x, mean(abs(x.Data - Gantrypirad(:,2)))];
error.theta = [error.theta, mean(abs(theta.Data - (-1 .*
Gantrypirad(:,3))))];

disp('Goodness of fit, x (m):')
disp(error.x(end));

disp('Goodness of fit, theta (rad):')
disp(error.theta(end));

%%%%%%%%%%%%%%%%%%%%%%%%

f = 6*pi;
c = 8;

sim('gantry')

hFig = figure(2);
set(hFig, 'Position', [0 0 1100 700]);
```

```
subplot(2,1,1);
hold on;
plot(x, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), Gantry6pirad(:, 2), 'LineWidth', 1.5);
title('c = 8');
xlabel('Time (s)');
ylabel('Cart x Position (m)');
legend('Simulation', 'Experimental');

subplot(2,1,2);
hold on;
plot(theta, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), -1 .* Gantry6pirad(:, 3), 'LineWidth', 1.5);
title('');
xlabel('Time (s)');
ylabel('Pendulum Angle (rad)');
legend('Simulation', 'Experimental');

error.x = [error.x, mean(abs(x.Data - Gantry6pirad(:,2)))];
error.theta = [error.theta, mean(abs(theta.Data - (-1 .*
Gantry6pirad(:,3))))];

disp('Goodness of fit, x (m):')
disp(error.x(end));

disp('Goodness of fit, theta (rad):')
disp(error.theta(end));

%%%%%%%%%%%%%%%%%%%%%%%%

f = 10*pi;
c = 12.5;

sim('gantry')

hFig = figure(3);
set(hFig, 'Position', [0 0 1100 700]);

subplot(2,1,1);
hold on;
plot(x, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), Gantry10pirad(:, 2), 'LineWidth', 1.5);
title('c = 12.5');
xlabel('Time (s)');
ylabel('Cart x Position (m)');
legend('Simulation', 'Experimental');

subplot(2,1,2);
hold on;
plot(theta, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), -1 .* Gantry10pirad(:, 3), 'LineWidth', 1.5);
```

```matlab
title('');
xlabel('Time (s)');
ylabel('Pendulum Angle (rad)');
legend('Simulation', 'Experimental');

error.x = [error.x, mean(abs(x.Data - Gantry10pirad(:,2)))];
error.theta = [error.theta, mean(abs(theta.Data - (-1 .*
Gantry10pirad(:,3))))];

disp('Goodness of fit, x (m):')
disp(error.x(end));

disp('Goodness of fit, theta (rad):')
disp(error.theta(end));

%%%%%%%%%%%%%%%%%%%%%%%%

f = 20*pi;
c = 15;

sim('gantry')

hFig = figure(4);
set(hFig, 'Position', [0 0 1100 700]);

subplot(2,1,1);
hold on;
plot(x, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), Gantry20pirad(:, 2), 'LineWidth', 1.5);
title('c = 15');
xlabel('Time (s)');
ylabel('Cart x Position (m)');
legend('Simulation', 'Experimental');

subplot(2,1,2);
hold on;
plot(theta, '--', 'LineWidth', 1.5);
plot(Gantrypirad(:,1), -1 .* Gantry20pirad(:, 3), 'LineWidth', 1.5);
title('');
xlabel('Time (s)');
ylabel('Pendulum Angle (rad)');
legend('Simulation', 'Experimental');

error.x = [error.x, mean(abs(x.Data - Gantry20pirad(:,2)))];
error.theta = [error.theta, mean(abs(theta.Data - (-1 .*
Gantry20pirad(:,3))))];

disp('Goodness of fit, x (m):')
disp(error.x(end));

disp('Goodness of fit, theta (rad):')
disp(error.theta(end));
```